Normal rings and normalization

Recall that an integral domain is <u>normal</u> if it is integrally closed in its field of fractions. Normality is closely connected to factoriality:

Prop: Let R be a ring. If R is a UFD, R is normal.

Pf: Consider $\frac{r}{s}$, w/ $r \in R$, $s \in R$. Assume r, s rel. prime.

Suppose $p\left(\frac{r}{s}\right) = \left(\frac{r}{s}\right)^n + a_1\left(\frac{r}{s}\right)^{n-1} + \dots + a_n = 0.$

$$\Rightarrow r^n = s\left(-a_1 r^{n-1} + \dots + a_n s^{n-1}\right)$$

$\Rightarrow s \mid r^n$, which contradicts relative primeness $\Rightarrow$ s is a unit in $R \Rightarrow \frac{r}{s} \in R$. $\square$

Cor: The only rational solutions to polynomials over $\mathbb{Z}$ are integers.

Cor: R is normal $\Rightarrow$ $R[x_1, \dots, x_n]$ is normal.

If $R \subseteq S$ are rings and $f \in R[x]$ monic, then having a root α is the same as $(x - \alpha) \mid f$. In fact, a more general statement holds:

Prop: If f factors in $S[x]$ as $f = gh$, g and h monic, then the coefficients of g and h are integral over R.

(Note that this also generalizes the statement that irreducible in $\mathbb{Z}[x] \Rightarrow$ irreducible in $\mathbb{Q}[x]$.)

Pf: Let $R[x]/(g) = R[\alpha_1]$, α_1 a root of g. Then using long division, we get $g = (x-\alpha_1)g_1$ over $R[\alpha_1]$. Repeating this, we get an extension ring T of S and elements α_i, β_j of T s.t.

$$g = \prod (x-\alpha_i) \text{ and } h = \prod (x-\beta_j) \text{ in } T[x].$$

So the α_i and β_j are integral over R so their coefficients are too. □

If R is an integral domain, then if f is monic and $f = gh$, then the leading coefficients are units (inverses of eachother), so we get the following:

Cor: If R is normal, then any monic irreducible polynomial $f \in R[x]$ is prime.

Pf: Let Q be the field of fractions of R. Then if $f = gh$ in $Q[x]$, $g, h \in R[x]$, so f is irreducible in $Q[x]$.

Since Q is a field, $Q[x]$ is a UFD. Thus $(f) \subseteq Q[x]$ is prime, and $R[x]/(f)$ is free over R.

$$\Rightarrow R[x]/(f) \longrightarrow Q \otimes_R R[x]/(f) = Q[x]/(f)$$

is just the direct sum of maps

$$R \to Q \otimes_R R = Q.$$

Thus it's injective, so $R[x]/(f)$ is an int. domain, so f is prime. □

An important property (geometrically, especially) of normalization is that it commutes w/ localization.

In particular, if we define a scheme by gluing together affine schemes (of the form spec R) along open sets, this says that we can "normalize" the scheme by first normalizing the affine schemes and then gluing those together, and the gluing will still work!

Prop: $R \subseteq S$ rings, U mult. closed subset of R. Let S' be the int. closure of R in S. Then $S'[U^{-1}]$ is the integral closure of $R[U^{-1}]$ in $S[U^{-1}]$.

Pf: An element of S integral over R is integral over $R[U^{-1}]$, so $S'[U^{-1}]$ is integral over $R[U^{-1}]$.

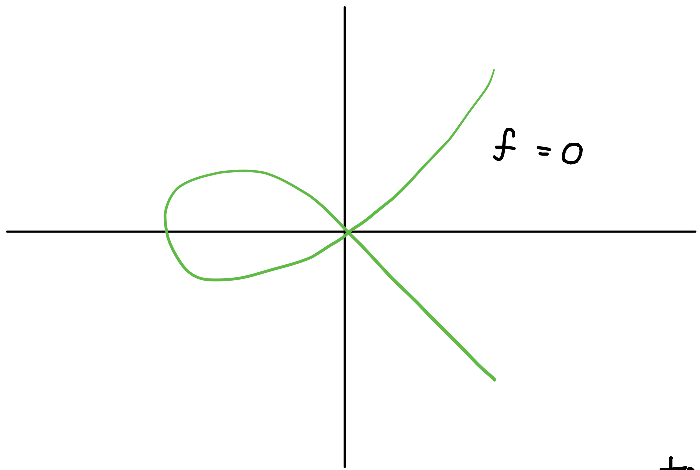
WTS that if $s/u \in S[U^{-1}]$ is integral over $R[U^{-1}]$, then sv is integral over R for some elt $v \in U$. i.e. $s/u \in S'[U^{-1}]$.

If $(s/u)^n + (r_1/u_1)(s/u)^{n-1} + \cdots = 0$, then clearing denominators by multiplying by $(u u_1 \cdots u_n)^n$, we get

$$(su_1 \cdots u_n)^n + r_1 u u_2 \cdots u_n (su_1 \cdots u_n)^{n-1} + \cdots = 0$$

$\Rightarrow$ $su_1 \cdots u_n$ is integral over r, so $\frac{s}{u} \in S'[u^{-1}]$. $\square$

Ex: Consider $f \in \mathbb{C}[x,y]$ defined $f = y^2 - x^2(x+1)$.



Consider the ring $R = \mathbb{C}[x,y]/(f)$.

Spec (R) is represented to the left. Note that R is exactly the polynomial functions restricted to the locus $f=0$. ($g = h$ on $f=0$ $g = h + af$, some a $\Leftrightarrow$ their image is equal in R)

R is not normal! y/x is integral over R:

If $p(t) = t^2 - (x+1) \in R[t]$, then $p(y/x) = (y/x)^2 - (x+1) = 0$.

Notice that if we try to evaluate y/x on $f=0$, it's defined away from the origin. If we take the limit, along 1 branch it's 1 and along the other it's -1. That is, the normalization "separates" the two branches.